and

$$N_1^2 - (2q+2-g)N_1+(q+1)^2-(q^2+1)g - 2qg^2 \leq 0,$$

from which the result follows.

For $g > \frac{1}{2}(q-\sqrt{q})$, Ihara's result is better than Serre's.

4. THE ESSENTIAL IDEA IN A PARTICULAR CASE

Let $\mathscr{C}$ be as in §2, but consider it as a curve over $\bar{K}$, the algebraic closure of $K = GF(q)$. Also suppose that $\mathscr{C}$ is embedded in the plane $PG(2,\bar{K})$ and let φ be the Frobenius map given by

$$P(x_o,x_1,x_2)\varphi = P(x_o^q,x_1^q,x_2^q)$$

where $P(x_o,x_1,x_2)$ is the point of the plane with coordinate vector (x_o,x_1,x_2). Then

$$\begin{aligned}\mathscr{C} &= V(F)\\ &= \{P(x_o,x_1,x_2) \mid F(x_o,x_1,x_2) = 0\}\end{aligned}$$

for some form F in $K[X_o,X_1,X_2]$. Also $\mathscr{C}\varphi = \mathscr{C}$ and the points of $\mathscr{C}$ rational over $GF(q)$ are exactly the fixed points of φ on $\mathscr{C}$.

For any non-singular point $P=P(x_o,x_1,x_2)$ the tangent T_p at P is

$$T_p = V\left(\frac{\partial F}{\partial x_o} X_o + \frac{\partial F}{\partial x_1} X_1 + \frac{\partial F}{\partial x_2} X_2\right) .$$

In affine coordinates,

$$T_p = V\left(\frac{\partial f}{\partial a} (x-a) + \frac{\partial f}{\partial b}(x-b)\right)$$

where $f(x,y) = F(x,y,1)$.

Instead of looking at fixed points of φ , let us look at the set of points such that $P\varphi \in T_p$. As $P \in T_p$, this set contains the GF(q)-rational points of $\mathscr{C}$. Let

$$h = (x^q - x)f_x + (y^q - y)f_y.$$

Then

$$h_x = (qx^{q-1} - 1)f_x + (x^q - x)f_{xx} + (y^q - y)f_{yx}$$
$$= -f_x + (x^q - x)f_{xx} + (y^q - y)f_{yx}$$

and

$$h_y = -f_y + (x^q - x)f_{xy} + (y^q - y)f_{yy}.$$

So V(h) and V(f) have a common tangent at any GF(q)-rational point of $\mathscr{C}$ that is non-singular. So, if N is the number of GF(q)-rational points of $\mathscr{C}$ and the degree of f is d, then Bézout's theorem implies, when f is not a component of h, that

$$\begin{aligned}(d+q-1)d &= \deg h \deg f \\ &= \text{sum of the intersection numbers at} \\ &\quad\ \text{the points of } V(f) \cap V(h) \\ &\geq 2N.\end{aligned}$$

Hence $N \leq \frac{1}{2}d(d+q-1)$.

Now, suppose that V(f) is a component of V(h), or equivalently that h=0 as a function an V(f). Therefore

$$(x^q - x)f_x/f_y + (y^q - y) = 0,$$
$$(x^q - x)\frac{dy}{dx} - (y^q - y) = 0.$$

Differentiating gives

$$(x^q-x)\,\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{d}{dx}(y^q-y) = 0$$

Remembering that $\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}$, we obtain that

$$(x^q-x)\,\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = 0.$$

Since $\frac{dy}{dx} = -f_x/f_y$, it follows that

$$\frac{d^2y}{dx^2} = -f_y^{-2}\,\{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2\}.$$

THEOREM 4.1: If $\frac{d^2y}{dx^2} \neq 0$, that is, $\mathscr{C}$ is not all inflexions and q is odd, then $N \leq \frac{1}{2}\,d(d+q-1)$.

In fact $\frac{d^2y}{dx^2} = 0$ can only occur when $\mathscr{C}$ is a line or the characteristic $p \leq d$. For example, when $f = x^{p^r+1} + y^{p^r+1}+1$, then $\mathscr{C}$ is all inflexions. A particular case of this phenomenon is the Hermitian curve $\mathscr{U}_{2,q} = V(X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1})$ when q is a square.

Since every curve of genus 3 can be embedded in the plane as a non-singular quartic, we can see how theorem 4.1 compares with Serre's bound for $N_q(3)$ and its actual value.

q	3	5	7	9	11	13	17	19
$2(q+3)$	12	16	20	24	28	32	40	44
$q+1+3[2\sqrt{q}]$	13	18	23	28	30	35	42	44
$N_q(3)$	10	16	20	28	28	32	40	44

Thus, for q odd with $q \leq 19$ and $q \neq 3$ or 9, the theorem gives the best possible result. A curve achieving $N_9(3)$ is $\mathscr{U}_{2,9}$.

5. WEIERSTRASS POINTS IN CHARACTERISTIC ZERO.

First consider the canonical curve $\mathscr{C}^{2g-2}$ of genus $g \geq 3$ in $PG(g-1,\mathbb{C})$. The Weierstrass points, W-points for short, are the points at which the osculating hyperplane has g coincident intersections. In this case, with w the number of W-points

$$w = g(g^2-1).$$

In any case,

$$2g + 2 \leq w \leq g(g^2-1)$$

with the lower bounded achieved only for hyperelliptic curves. A curve of genus $g > 1$ is hyperelliptic if it has a linear series γ^1_2 (a 2-sheeted covering) on it; for example, a plane quartic with a double point. It has equation

$$y^2 = f(x)$$

with genus $g = [\frac{1}{2}(d-1)]$ where $d = \deg f$.

Consider the case g=3 of the canonical curve $\mathscr{C}^4$, a non-singular plane quartic. The W-points are the 24 inflexions. We note that